Approximation of Stochastic Partial Differential Equations by a Kernel-based Collocation Method

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Parabolic Stochastic Equations $\implies$ Elliptic Stochastic Equations

Here, we only consider the simple high-dimensional elliptic SPDE

$$\begin{cases} 
\Delta u = f + \xi, & \text{in } \mathcal{D} \subset \mathbb{R}^d, \\
u = 0, & \text{on } \partial\mathcal{D},
\end{cases}$$

where

- $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian operator,
- suppose that $u \in$ Sobolev space $H^m(\mathcal{D})$ with $m > 2 + d/2$ a.s.,
- $f : \mathcal{D} \to \mathbb{R}$ is a deterministic function,
- $\xi : \mathcal{D} \times \Omega_\xi \to \mathbb{R}$ is a Gaussian field with mean zero and covariance kernel $W : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ defined on a probability space $(\Omega_\xi, \mathcal{F}_\xi, \mathbb{P}_\xi)$, i.e.,

$$E(\xi_x) = 0, \quad Cov(\xi_x, \xi_y) = W(x, y).$$
The proposed numerical method for solving a parabolic SPDE can be described as follows:

1. We choose a reproducing kernel

\[ K : D \times D \rightarrow \mathbb{R} \]

whose reproducing kernel Hilbert space \( H_K(D) \) is embedded into \( H^m(D) \).

\[ \begin{align*}
\text{Noise Covariance Kernel } W & \quad \rightarrow \quad \text{Smoothness of Exact Solution } u \\
\downarrow & \quad \downarrow \\
\text{Convergent Rates} & \quad \leftarrow \quad \text{Reproducing Kernel } K
\end{align*} \]
We simulate the Gaussian field with covariance structure $W$ at a finite collection of predetermined collocation points

$$X_D := \{x_1, \cdots, x_N\} \subset D, \quad X_{\partial D} := \{x_{N+1}, \cdots, x_{N+M}\} \subset \partial D,$$

i.e.,

$$y_j := f(x_j) + \xi_{x_j}, \quad j = 1, \cdots, N, \quad y_{N+j} := 0, \quad j = 1, \cdots, M,$$

and

$$\xi := (\xi_{x_1}, \cdots, \xi_{x_N}) \sim \mathcal{N}(0, W), \quad W := (W(x_j, x_k))_{j,k=1}^{N,N}.$$

We also let the random vector

$$y_\xi := (y_1, \cdots, y_{N+M})^T.$$
We also define its integral-type kernel

\[ K(x, y) := \int_D K(x, z)K(y, z)dz, \quad K \in \mathcal{H}^{m,m}(D \times D). \]

The kernel-based collocation solution is written as

\[
\begin{align*}
    u(x) \approx \hat{u}(x) & := \sum_{k=1}^{N} c_k \Delta_2^* K(x, x_k) + \sum_{k=1}^{M} c_{N+k} \Delta^* K(x, x_{N+k}), \\
    \text{where the unknown random coefficients} \quad c & := (c_1, \cdots, c_{N+M})^T \\
    \text{are obtained by solving a random system of linear equations, i.e.,} \\
    \Delta^* c & = y_\xi.
\end{align*}
\]
Advantages

- The **kernel-based** collocation method is a **meshfree** approximation method. It does not require an underlying triangular mesh as the **Galerkin finite element** method does.

- The **kernel-based** collocation method can be applied to a **high-dimensional** domain \( D \) with **complex** boundary \( \partial D \).

- To obtain the **truncated** Gaussian noise \( \xi^n \) for the **finite element** method, it is difficult for us to compute the **eigenvalues** and **eigenfunctions** of the noise covariance kernel \( W \). For the **kernel-based** collocation method we need not worry about this issue.

- Once the **reproducing kernel** is fixed, the error of the collocation solution only depends on the **collocation points**.
Given a finite element basis $\phi$, we shall compute the right-hand side for the Galerkin finite element methods.

- **Popular Methods:**

$$
\int_D \xi_x \phi(x) \, dx \approx \int_D \xi^n_x \phi(x) \, dx = \sum_{k=1}^n \zeta_k \int_D \sqrt{\lambda_k} e_k(x) \phi(x) \, dx,
$$

where the truncated Gaussian noise

$$
\xi_x \approx \xi^n_x = \sum_{k=1}^n \zeta_k \sqrt{\lambda_k} e_k(x), \quad \zeta_1, \ldots, \zeta_n \sim \text{i.i.d.}\, \mathcal{N}(0, 1),
$$

and

$$
W(x, y) \approx W^n(x, y) = \sum_{k=1}^n \lambda_k e_k(x) e_k(y).
$$
Monte Carlo Methods:
For each fixed sample path $\omega \in \Omega_\xi$, $\xi_x(\omega)$ is a function defined on $\mathcal{D}$. However, we do not know its exact form. We can only use Monte Carlo methods to approximate the right-hand side, i.e.,

$$\int_{\mathcal{D}} \xi_x \phi(x) dx \approx \sum_{j=1}^{N} \xi_{x_j} \phi(x_j).$$

Kernel-based Methods:

$$\xi_x \approx \hat{\xi}_x := w(x)^T W^{-1} \xi,$$

where

$$w(x) := (W(x, x_1), \cdots, W(x, x_N))^T, \quad W := (W(x_j, x_k))_{j,k=1}^{N,N}.$$
According to [Cialenco, Fasshauer and Ye 2011 SPDE, Theorem 3.1], for a given $\mu \in H_K(D)$, there exists a probability measure $\mathbb{P}^\mu$ defined on

$$(\Omega_K, \mathcal{F}_K) = (H_K(D), \mathcal{B}(H_K(D)))$$

such that the stochastic fields $\Delta S, S$ given by

$$\Delta S_x(\omega) = \Delta S(x, \omega) := (\Delta \omega)(x), \quad x \in D, \quad \omega \in \Omega_K = H_K(D),$$

$$S_x(\omega) = S(x, \omega) := \omega(x), \quad x \in D \cup \partial D, \quad \omega \in \Omega_K = H_K(D),$$

are Gaussian with means $\Delta \mu$, $\mu$ and covariance kernels $\Delta_1 \Delta_2 K, K$ defined on $(\Omega_K, \mathcal{F}_K, \mathbb{P}^\mu)$, respectively.

For any fixed $z \in \mathbb{R}$, we let

$$\mathcal{E}_x(z) := \{ \omega \in \Omega_K : \omega(x) = z \} = \{ \omega \in \Omega_K : S_x(\omega) = z \}.$$
[Cialenco, Fasshauer and Ye 2011 SPDE, Corollary 3.2], shows that the random vector

\[ S := (\Delta S_{x_1}, \cdots, \Delta S_{x_N}, S_{x_{N+1}}, \cdots, S_{x_{N+M}}) \sim \mathcal{N}(m^\mu, K), \]

where

\[ m^\mu := (\Delta \mu(x_1), \cdots, \Delta \mu(x_N), \mu(x_{N+1}), \cdots, \mu(x_{N+M}))^T \]

\[ K := \begin{pmatrix} \Delta_1 \Delta_2^* K(x_j, x_k)_{j,k=1}^{N,N} & (\Delta_1^* K(x_j, x_{N+k}))_{j,k=1}^{N,M} \\ (\Delta_2^* K(x_{N+j}, x_k))_{j,k=1}^{M,N} & (K(x_{N+j}, x_{N+k}))_{j,k=1}^{M,M} \end{pmatrix}. \]

For any given \( y = (y_1, \cdots, y_{N+M})^T \in \mathbb{R}^{N+M} \), we let

\[ \mathcal{E}_X(y) := \{ \omega \in \Omega_K : \Delta \omega(x_1) = y_1, \ldots, \omega(x_{N+M}) = y_{N+M} \} \]

\[ = \{ \omega \in \Omega_K : S(\omega) = y \}. \]
For each fixed $x \in D$ and $\omega_2 \in \Omega_\xi$, we obtain the "optimal" estimator

$$u(x, \omega_2) \approx \hat{u}(x, \omega_2) = \arg\max_{z \in \mathbb{R}} \sup_{\mu \in H_K(D)} \mathbb{P}_\xi^\mu \left( \mathcal{E}_x(z) \times \Omega_\xi \mid \mathcal{E}_x \left( y_\xi(\omega_2) \right) \right),$$

$$= \arg\max_{z \in \mathbb{R}} \sup_{\mu \in H_K(D)} \mathbb{P}_\xi^\mu \left( S_x = z \mid S = y_\xi(\omega_2) \right),$$

$$= \arg\max_{z \in \mathbb{R}} \sup_{\mu \in H_K(D)} \rho_x^\mu(z \mid y_\xi(\omega_2)),$$

$$= k(x)^T K^{-1} y_\xi(\omega_2),$$

where $k(x) := (\Delta_2^* K(x, x_1), \cdots, \Delta_2^* K(x, x_{N+M}))^T$ and

$$\Omega_{K\xi} := \Omega_K \times \Omega_\xi, \quad \mathcal{F}_{K\xi} := \mathcal{F}_K \otimes \mathcal{F}_\xi, \quad \mathbb{P}_\xi^\mu := \mathbb{P}^\mu \otimes \mathbb{P}_\xi,$$

so that $\Delta S$, $S$ and $\xi$ can be extended to the product space while preserving the original probability distributional properties.
Error Bound Analysis

For any $\epsilon > 0$, we define

$$E_x^\epsilon := \left\{ \omega_1 \times \omega_2 \in \Omega_K \times \Omega_\xi : |\omega_1(x) - \hat{u}(x, \omega_2)| \geq \epsilon, \right\}$$

s.t. $\Delta \omega_1(x_1) = y_1(\omega_2), \ldots, \omega_1(x_{N+M}) = y_{N+M}(\omega_2)$$.$

Let the fill distance

$$h_x := \sup_{x \in D} \min_{1 \leq j \leq N+M} \|x - x_j\|_2.$$
We can deduce that

$$\sup_{\mu \in H_K(D)} \mathbb{P}^\mu_\xi(\mathcal{E}^\epsilon_x) = \mathcal{O}\left(\frac{h^m x^{-d/2}}{\epsilon}\right),$$

where $m$ is the order of the Sobolev space corresponded to the exact solution of the SPDE.

Since $|u(x, \omega_2) - \hat{u}(x, \omega_2)| \geq \epsilon$ if and only if $u \in \mathcal{E}^\epsilon_x$, we have

$$\sup_{\mu \in H_K(D)} \mathbb{P}^\mu_\xi (\|u - \hat{u}\|_{L_\infty(D)} \geq \epsilon) \leq \sup_{\mu \in H_K(D), x \in D} \mathbb{P}^\mu_\xi (\mathcal{E}^\epsilon_x) \to 0,$$

when $h_x \to 0.$
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Let the domain
\[ D := (0, 1)^2 \subset \mathbb{R}^2. \]

We choose the deterministic function
\[ f(x) := -2\pi^2 \sin(\pi x_1) \sin(\pi x_2) - 8\pi^2 \sin(2\pi x_1) \sin(2\pi x_2), \]
and the covariance kernel of the Gaussian noise \( \xi \) to be
\[ W(x, y) := 4\pi^4 \sin(\pi x_1) \sin(\pi x_2) \sin(\pi y_1) \sin(\pi y_2) \\ + 16\pi^4 \sin(2\pi x_1) \sin(2\pi x_2) \sin(2\pi y_1) \sin(2\pi y_2). \]

Then the exact solution of the above elliptic SPDE has the form
\[ u(x) := \sin(\pi x_1) \sin(\pi x_2) + \sin(2\pi x_1) \sin(2\pi x_2) \\ + \zeta_1 \sin(\pi x_1) \sin(\pi x_2) + \frac{\zeta_2}{2} \sin(2\pi x_1) \sin(2\pi x_2), \]
where \( \zeta_1, \zeta_2 \sim \text{i.i.d. } \mathcal{N}(0, 1). \)
For the collocation methods, we use the $C^4$-Matérn function with shape parameter $\theta > 0$

$$g_\theta(r) := (3 + 3\theta r + \theta^2 r^2)e^{-\theta r}, \quad r > 0,$$

to construct the reproducing kernel (Sobolev-spline kernel)

$$K_\theta(x, y) := g_\theta(\|x - y\|_2).$$

According to [Fasshauer and Ye 2011 Distributional Operators, Fasshauer and Ye 2011 Differential and Boundary Operators], we can deduce that

$$\mathcal{H}_K(\mathcal{D}) \cong \mathcal{H}^{3+1/2}(\mathcal{D}) \subset C^2(\mathcal{D}).$$
Numerical Examples

Stochastic Laplace’s Equations

Figure: $N = 65$, $M = 28$ and $\theta = 0.9$
Figure: Convergence of Mean and Variance

- Mean, $\theta = 0.9$
- Mean, $\theta = 1.9$
- Mean, $\theta = 2.9$
- Variance, $\theta = 0.9$
- Variance, $\theta = 1.9$
- Variance, $\theta = 2.9$
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