Linear tensor product problems in (anti-) symmetric Hilbert spaces

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Some motivation

Yserentant '10: complexity results using antisymmetry properties

\[ \Rightarrow \text{(anti-) symmetry} \text{ as a new kind of a priori knowledge} \]

for tractability studies

(easiest setting: tensor product problems between Hilbert spaces)
Linear tensor product problems

Let

- $H_1 \ldots$ (infinite dimensional) separable Hilbert space, e.g., $H_1 = W^1_2([0, 1])$
- $G_1 \ldots$ arbitrary Hilbert space, e.g., $G_1 = L_2([0, 1])$
- $S_1 : H_1 \to G_1 \ldots$ compact linear operator, e.g., $S_1 = \text{id} : W^1_2 \hookrightarrow L_2$
- $H_d = H_1 \otimes \ldots \otimes H_1 \ldots d$-fold tensor product space ($d \in \mathbb{N}$),

\[
\left\langle \bigotimes_{i=1}^{d} f_i, \bigotimes_{i=1}^{d} g_i \right\rangle_{H_d} = \prod_{i=1}^{d} \left\langle f_i, g_i \right\rangle_{H_1} \quad \text{for} \quad f_i, g_i \in H_1
\]

analogously $G_d = G_1 \otimes \ldots \otimes G_1$
- $S_d = S_1 \otimes \ldots \otimes S_1 : H_d \to G_d \ldots$ tensor product operator,

\[
S_d \left( \bigotimes_{i=1}^{d} f_i \right) = \bigotimes_{i=1}^{d} S_1(f_i).
\]
Then $S_d$ is also compact and we know a linear algorithm $A_{n,d}'$, using $n \in \mathbb{N}$ continuous linear functionals, which minimizes the worst case error of $A_{n,d}$:

$$e^{\text{wor}}(A_{n,d}) = \sup_{\|f\|_{H_d} \leq 1} \|S_d(f) - A_{n,d}(f)\|_{G_d}.$$ 

Moreover, we know exact formulae for

$$e(n, d) = \inf_{A_{n,d}} e^{\text{wor}}(A_{n,d}) \quad (n\text{-th minimal error of } S_d),$$

and

$$\varepsilon_{\text{init}}^d = e(0, d) = \|S_d\|_{H_d \rightarrow G_d} = \|S_1\|^d \quad (\text{initial error of } S_d),$$

as well as estimates on the information complexity:

$$n(\varepsilon, d) = \min \{n \in \mathbb{N}_0 \mid e(n, d) \leq \varepsilon\}, \quad \varepsilon > 0, d \in \mathbb{N}.$$
Problem \( S = \{ S_d \mid d \in \mathbb{N} \} \) is called

- **polynomially tractable** w.r.t. the absolute error criterion \((PT_{\text{abs}}) : \iff \exists C, p, q \geq 0 \text{ such that}\)

\[
n(\varepsilon, d) \leq C \cdot \varepsilon^{-p} \cdot d^q \quad \forall \varepsilon \in (0, \varepsilon_d^{\text{init}}] \quad \forall d \in \mathbb{N}
\]

- **strongly polynomially tractable** w.r.t. the absolute error criterion \((SPT_{\text{abs}}) : \iff PT_{\text{abs}} \text{ with } q = 0\)

Similar definitions for the *normalized* error criterion.
Known results

Main tool for tractability studies:

SVD (Singular value decomposition) of $S_d$, i.e. study of eigenpairs
\[ \{(\lambda_{d,j}, e_{d,j}) \mid j \in \mathbb{N}^d\} \]
of
\[ W_d = S_d^\dagger S_d : H_d \to H_d, \quad d \in \mathbb{N}, \]
in terms of \[ \{(\lambda_m, e_m) \mid m \in \mathbb{N}\} \] (eigenpairs of $W_1$)

Theorem (see Novak & Woźniakowski '08)

\[ SPT_{\text{abs}} \iff PT_{\text{abs}} \]
\[ \iff \lambda_1 < 1 \quad \text{and} \quad \lambda_m \in \mathcal{O}(m^{-r}) \quad \text{as} \quad m \to \infty \quad \text{for some} \quad r > 0 \]
\[ \iff \lambda_1 < 1 \quad \text{and} \quad \lambda = (\lambda_m)_{m \in \mathbb{N}} \in \ell_\tau \quad \text{for some} \quad \tau > 0 \]

\[ \text{NEVER tractable w.r.t. the normalized error (except trivial cases)} \]
Problem: case $\lambda_1 = \|S_1\| = 1$ important for applications
$(\lambda_1 < 1$ implies $\varepsilon_d^{\text{init}} = \|S_1\|^d \rightarrow 0$ exponentially fast$)$

Approaches:
- relaxing the error definitions, e.g., average case error
- exploiting further properties, such as
  - weights in the norm of $H_d$, or
  - (anti-) symmetry,

  to shrink the unit ball
Definition of (anti-) symmetry

- For every $d \in \mathbb{N}$ fix a set of coordinates $\emptyset \neq l_d \subseteq \{1, \ldots, d\}$
  $\implies \# l_d$ describes the amount of (anti-) symmetry in dimension $d$

- $f: [0, 1]^d \rightarrow \mathbb{R}$ symmetric w.r.t. the subset $l_d$ :

  $f(x) = f(\pi(x)) \quad \forall x \in [0, 1]^d \quad \forall$ permutations $\pi$ on $l_d$

- $f: [0, 1]^d \rightarrow \mathbb{R}$ antisymmetric w.r.t. the subset $l_d$ :

  $f(x) = \text{sign}(\pi) \cdot f(\pi(x)) \quad \forall x \in [0, 1]^d \quad \forall \pi$

- $l_d = \{1, \ldots, d\} \implies f$ fully (anti-) symmetric
In the antisymmetric case:

\[ x_i = x_j \text{ for } i \neq j \text{ in } I_d \quad \implies \quad f(x_1, \ldots, x_d) = 0, \]

whereas this does not hold in the symmetric case in general!  
\[ \implies \text{antisymmetry is a stronger condition than symmetry} \]
Task: study tractability of $S_d$ restricted to subspaces of (anti-) symmetric functions in $H_d$

There exist orthogonal projections $P_{I_d}$ onto these subspaces. The problem operator $S_d$ commutes with $P_{I_d}$.

Using standard techniques this observation leads to an optimal algorithm for (anti-) symmetric problems, where the error can be expressed in terms of the univariate eigenvalues $\lambda = (\lambda_m)_{m \in \mathbb{N}}$. 
Tractability (symmetric case)

**Theorem (absolute error)**

\[
\text{PT}_{\text{abs}} \iff \lambda = (\lambda_m)_{m \in \mathbb{N}} \in \ell_{\tau} \text{ for some } \tau > 0 \text{ and }
\]

- \(\lambda_1 < 1\), or
- \(\lambda_1 = 1\) and \((d - \#I_d) \in \mathcal{O}(\ln d)\).

\[
\text{SPT}_{\text{abs}} \iff \lambda \in \ell_{\tau} \text{ for some } \tau > 0 \text{ and }
\]

- \(\lambda_1 < 1\), or
- \(\lambda_1 = 1 > \lambda_2\) and \((d - \#I_d) \in \mathcal{O}(1)\).

In particular, for the **fully** symmetric problem:

\[
\text{PT}_{\text{abs}} \iff \lambda \in \ell_{\tau} \text{ and } \lambda_1 \leq 1
\]
Theorem (absolute error)

- If $\lambda_1 < 1$ then

$$SPT_{\text{abs}} \iff PT_{\text{abs}} \iff \lambda = (\lambda_m)_{m \in \mathbb{N}} \in \ell_{\tau} \text{ for some } \tau > 0.$$  

- If $\lambda_1 \geq 1$ and $\# I_d$ grows linearly with the dimension then the same equivalences hold true.

In particular, for the **fully** antisymmetric problem:

$$PT_{\text{abs}} \iff SPT_{\text{abs}} \iff \lambda \in \ell_{\tau}$$
Summary

In this talk we...

- ...considered linear tensor product problems defined between Hilbert spaces
- ...introduced an essentially new kind of a priori knowledge, which appears in practice
- ...obtained optimal linear algorithms using information from $\Lambda^{\text{all}}$.

We found:

- **unrestricted** setting $\implies$ complete knowledge about complexity issues in terms of univariate singular values, but **bad tractability behavior**
- additional **(anti-) symmetry** conditions $\implies$ **significant improvements** of the complexity only depending on the amount of a priori knowledge


Thank you for your attention!