ML-QMC convergence analysis for stochastic and parametric operator equations in infinite dimension

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Parametric Elliptic Equations

Model parametric elliptic Dirichlet problem in bounded Lipschitz domain $D \subset \mathbb{R}^d$:

$$-\nabla \cdot (a(\boldsymbol{x},\boldsymbol{y})\,\nabla u(\boldsymbol{x},\boldsymbol{y})) \;=\; f(\boldsymbol{x}) \quad \text{in} \quad D \subset \mathbb{R}^d\,, \quad u(\boldsymbol{x},\boldsymbol{y}) \;=\; 0 \quad \text{on} \quad \partial D\,. \tag{1}$$

$$a(\boldsymbol{x},\boldsymbol{y}) \;=\; \bar{a}(\boldsymbol{x}) + \sum_{j\geq 1} y_j\,\psi_j(\boldsymbol{x})\,, \qquad \boldsymbol{x} \in D\,, \quad \boldsymbol{y} \in U\,, \tag{2}$$

$$\boldsymbol{y} = (y_1, y_2, \ldots) \in (-\tfrac{1}{2}, \tfrac{1}{2})^{\mathbb{N}} \;=:\; U\,.$$

(A1) $\bar{a} \in L^\infty(D)$ and $\sum_{j\geq 1} \|\psi_j\|_{L^\infty(D)} < \infty$.

(A2) There exist $a_{\max}$ and $a_{\min}$ such that $0 < a_{\min} \leq a(\boldsymbol{x},\boldsymbol{y}) \leq a_{\max}$ for all $\boldsymbol{x} \in D$ and $\boldsymbol{y} \in U$.

(A3) There exists $p \in (0,1]$ such that $\sum_{j\geq 1} \|\psi_j\|^p_{L^\infty(D)} < \infty$.

(A4) With $\|v\|_{W^{1,\infty}(D)} := \max\{\|v\|_{L^\infty(D)}, \|\nabla v\|_{L^\infty(D)}\}$, $\bar{a} \in W^{1,\infty}(D)$ and $\sum_{j\geq 1} \|\psi_j\|_{W^{1,\infty}(D)} < \infty$.

(A5) $\|\psi_1\|_{L^\infty(D)} \geq \|\psi_2\|_{L^\infty(D)} \geq \cdots$.

(A6) The domain D is a convex and bounded polyhedron with plane faces.

(A7) For p as in **(A3)**, there exists $q \in [p,1]$ such that $\sum_{j\geq 1} \|\psi_j\|^q_{W^{1,\infty}(D)} < \infty$.

Multi-Level Quasi Monte Carlo Finite Element Approximation

Linear functional $G(\cdot) : H_0^1(D) :\to \mathbb{R}$. Goal: approximate

$$I(G(u)) := \int_U G(u(\cdot,\boldsymbol{y}))\,\mathrm{d}\boldsymbol{y} := \lim_{s\to\infty}\int_{(-\frac{1}{2},\frac{1}{2})^s} G(u(\cdot,(y_1,\ldots,y_s,0,0,\ldots)))\,\mathrm{d}y_1\cdots\mathrm{d}y_s \ .$$

Single level QMC (Kuo, Sloan & CS):

(i) (dimension truncation) truncate the infinite sum $a(\boldsymbol{x},\boldsymbol{y})$ to s terms,

(ii) approximate the solution of the truncated PDE problem using a FEM with mesh size h, and

(iii) approximate the integral using a QMC (equal-weight) quadrature rule with N points in s dimensions.

(single level) QMC-FEM:

$$Q_{s,N}(G(u_h^s)) := \frac{1}{N}\sum_{i=1}^{N} G\big(u_h^s(\cdot,\boldsymbol{y}^{(i)})\big) \ , \tag{3}$$

where u_h^s denotes the FE solution of the dimension-truncated PDE problem, $\boldsymbol{y}^{(1)},\ldots,\boldsymbol{y}^{(N)} \in [-\frac{1}{2},\frac{1}{2}]^s$.
Multi Level QMC-FEM:

$$Q_*^L(G(u)) := \sum_{\ell=0}^{L} Q_{s_\ell,N_\ell}\Big(G\big(u_{h_\ell}^{s_\ell} - u_{h_{\ell-1}}^{s_{\ell-1}}\big)\Big) \ . \tag{4}$$

Finite Element Discretization

Parameter-dependent weak formulation: for $\boldsymbol{y} \in U$, find

$$u(\cdot, \boldsymbol{y}) \in V \ : \ \int_D a(\boldsymbol{x}, \boldsymbol{y}) \, \nabla u(\boldsymbol{x}, \boldsymbol{y}) \cdot \nabla v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \ = \ \int_D f(\boldsymbol{x}) \, v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \qquad \forall v \in V \ . \tag{5}$$

Here

$$b(\boldsymbol{y}; w, v) \ := \ \int_D a(\boldsymbol{x}, \boldsymbol{y}) \, \nabla w(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \ , \qquad \forall w, v \in V \ . \tag{6}$$

(A2) $\Longrightarrow$ continuous and coercive on $V \times V$:

$$\forall \boldsymbol{y} \in U \ , \forall v, w \in V : \quad b(\boldsymbol{y}; v, v) \ \geq \ a_{\min} \, \|v\|_V^2 \quad \text{and} \quad |b(\boldsymbol{y}; v, w)| \ \leq \ a_{\max} \, \|v\|_V \, \|w\|_V \ .$$

Theorem

(a) Under Assumptions $(\mathbf{A1})$ and $(\mathbf{A2})$, for every $f \in V^*$ and every $\boldsymbol{y} \in U$, there exists a unique solution $u(\cdot, \boldsymbol{y}) \in V$ of the parametric weak problem (5), which satisfies

$$\|u(\cdot, \boldsymbol{y})\|_V \ \leq \ \frac{\|f\|_{V^*}}{a_{\min}} \ . \tag{7}$$

(b) If, in addition, $f \in L^2(D)$ and if Assumption $(\mathbf{A4})$ holds, then there exists a constant $C > 0$ such that for every $\boldsymbol{y} \in U$,

$$\|u(\cdot, \boldsymbol{y})\|_Z \leq C \|f\|_{L^2(D)} \ . \tag{8}$$

Finite Element Discretization

Theorem (dimension truncation)
Define

$$b_j := \frac{\|\psi_j\|_{L^\infty(D)}}{a_{\min}} , \qquad j \geq 1 . \tag{9}$$

Under Assumptions $(\mathbf{A1})$ and $(\mathbf{A2})$, for every $f \in V^*$, every $\boldsymbol{y} \in U$ and every $s \in \mathbb{N}$, the solution $u^s(\boldsymbol{x},\boldsymbol{y}) = u(\boldsymbol{x},(\boldsymbol{y}_{\{1:s\}};0))$ of the truncated parametric weak problem (5) satisfies the bound

$$\|u(\cdot,\boldsymbol{y}) - u^s(\cdot,\boldsymbol{y})\|_V \leq \frac{\|f\|_{V^*}}{2\,a_{\min}} \sum_{j\geq s+1} b_j ,$$

If, moreover, $(\mathbf{A3})$ and $(\mathbf{A5})$ hold, then

$$\sum_{j\geq s+1} b_j \leq \begin{cases} \min(\frac{1}{1/p-1},1) s^{-(1/p-1)} \left(\sum_{j\geq s+1} b_j^p\right)^{1/p} & \text{if } p<1 , \\ \frac{c}{\eta}\, s^{-\eta} & \text{if } p=1 . \end{cases} \tag{10}$$

Finite Element Discretization

$\{V_h\}_h$ dense, one-parameter family of subspaces $V_h \subset V$ of dimensions $M_h < \infty$. e.g continuous, piecewise-linear finite elements on a sequence of regular, simplicial meshes $\mathcal{T}_h$ in D obtained from an initial, regular triangulation $\mathcal{T}_0$ of D by recursive, uniform bisection of simplices.

$v \in Z \Longrightarrow v \in V \cap H^2(D)$, and ex. $C > 0$ such that, as $h \to 0$, holds the *approximation property*

$$\inf_{v_h \in V_h} \|v - v_h\|_V \;\le\; C\, h\, \|v\|_Z \; .$$

parametric FE approximation $u_h(\cdot, \boldsymbol{y})$: for $f \in V^*$ and $\boldsymbol{y} \in U$, find

$$u_h(\cdot, \boldsymbol{y}) \in V_h : \quad b(\boldsymbol{y}; u_h(\cdot, \boldsymbol{y}), v_h) \;=\; \langle f, v_h \rangle \qquad \forall v_h \in V_h \; . \tag{11}$$

Theorem
$(\mathbf{A1})$, $(\mathbf{A2})$, $(\mathbf{A4})$, and $(\mathbf{A6}) \Longrightarrow \forall f \in V^*, \forall \boldsymbol{y} \in U$:

$$\|u_h(\cdot, \boldsymbol{y})\|_V \;\le\; \frac{\|f\|_{V^*}}{a_{\min}} \; .$$

Moreover, $\forall f \in L^2(D)$ $, \forall \boldsymbol{y} \in U$,

$$\|u(\cdot, \boldsymbol{y}) - u_h(\cdot, \boldsymbol{y})\|_V \;\le\; C\, h\, \|u(\cdot, \boldsymbol{y})\|_Z \;\le\; C\, h\, \|f\|_{L^2(D)} \; , \tag{12}$$

where the constant $C > 0$ is independent of h and $\boldsymbol{y}$.

Finite Element Discretization

Aubin-Nitsche: $(\mathbf{A1})$, $(\mathbf{A2})$, $(\mathbf{A4})$, and $(\mathbf{A6}) \Longrightarrow \forall f \in L^2(D)\ , \forall G(\cdot) \in L^2(D)\ , \forall \boldsymbol{y} \in U$, the FE approximations $G(u_h(\cdot, \boldsymbol{y}))$ satisfy the asymptotic convergence estimate as $h \to 0$

$$|G(u(\cdot, \boldsymbol{y})) - G(u_h(\cdot, \boldsymbol{y}))| \le C\, h^2\, \|f\|_{L^2(D)}\, \|G(\cdot)\|_{L^2(D)}\ , \tag{13}$$

where the constant $C > 0$ is independent of h and $\boldsymbol{y}$.
Use QMC-Quadrature for the integrand function

$$F(\boldsymbol{y}) := G(u_h(\cdot, \boldsymbol{y}))\ , \qquad \boldsymbol{y} \in U\ .$$

QMC-Approximation

$$I_s(F) \;:=\; \int_{[-\frac{1}{2},\frac{1}{2}]^s} F(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\;.$$

One realization of an N-point shifted lattice rule takes the form

$$Q_{s,N}(\boldsymbol{\Delta};F) \;:=\; \frac{1}{N}\sum_{i=1}^{N} F\left(\mathrm{frac}\left(\frac{i\boldsymbol{z}}{N}+\boldsymbol{\Delta}\right)-\left(\tfrac{1}{2},\dots,\tfrac{1}{2}\right)\right)\;,$$

$\boldsymbol{z}\in\mathbb{Z}^s$ *generating vector*, $\boldsymbol{\Delta}$ *random shift*, $\sim\mathcal{U}([0,1]^s)$.

Theorem

Let $s,N\in\mathbb{N}$ be given, with N a prime number, and assume $F\in\mathcal{W}_{s,\boldsymbol{\gamma}}$ for a particular choice of weights $\boldsymbol{\gamma}=(\gamma_{\mathfrak{u}})_{\mathfrak{u}}$. Then a randomly shifted lattice rule can be constructed using a component-by-component algorithm such that the root-mean-square error satisfies, for all $\lambda\in(1/2,1]$,

$$\sqrt{\mathbb{E}\left[|I_s(F)-Q_{s,N}(\cdot;F)|^2\right]}\le\left(\sum_{\emptyset\neq\mathfrak{u}\subseteq\{1:s\}}\gamma_{\mathfrak{u}}^{\lambda}\,[\rho(\lambda)]^{|\mathfrak{u}|}\right)^{1/(2\lambda)}(N-1)^{-1/(2\lambda)}\,\|F\|_{\mathcal{W}_{s,\boldsymbol{\gamma}}}\;,$$

where $\mathbb{E}[\cdot]$ denotes the expectation with respect to the random shift which is uniformly distributed over $[0,1]^s$,

$$\rho(\lambda)\;:=\;\frac{2\zeta(2\lambda)}{(2\pi^2)^{\lambda}}+\frac{1}{12^{\lambda}}\;,\qquad\zeta(x)=\sum_{k=1}^{\infty}k^{-x}\;.\qquad(14)$$

Multi-Level QMC-FE Approximation

Let

$$h_\ell = 2^{-\ell} h_0 \quad \text{for} \quad \ell = 0, 1, 2, \ldots \ . \tag{15}$$

Assume sequence $\{s_\ell\}_{\ell \geq 0}$ of active dimensions is nondecreasing, i.e.,

$$s_0 \leq s_1 \leq \cdots s_\ell \leq \cdots \ , \tag{16}$$

$$V_\ell \equiv V_{h_\ell} \ , \quad \mathcal{T}_\ell \equiv \mathcal{T}_{h_\ell} \ , \quad Q_\ell \equiv Q_{s_\ell, N_\ell} \ , \quad I_\ell \equiv I_{s_\ell} \ , \quad u_\ell \equiv u_{h_\ell}^{s_\ell} \ , \quad M_\ell \equiv M_{h_\ell} \ . \tag{17}$$

ML-QMC-FE approximation of $I(G(u))$ is given by

$$Q_*^L(\boldsymbol{\Delta}_*; G(u)) := \sum_{\ell=0}^{L} Q_\ell(\boldsymbol{\Delta}_\ell; G(u_\ell - u_{\ell-1})) \ , \tag{18}$$

where $\boldsymbol{\Delta}_* := (\boldsymbol{\Delta}_0, \ldots, \boldsymbol{\Delta}_L)$ is the "compound shift".
In practice, expectation $\mathbb{E}[\cdot]$ over all random shifts sampled by MC:

$$Q^L(G(u)) := \sum_{\ell=0}^{L} \frac{1}{m_\ell} \sum_{i=1}^{m_\ell} Q_\ell(\boldsymbol{\Delta}_{\ell,i}; G(u_\ell - u_{\ell-1})) \ . \tag{19}$$

ML-QMC-FE Error Analysis

$$I(G(u)) - Q_*^L(\boldsymbol{\Delta}_*; G(u)) \;=\; I(G(u)) - \sum_{\ell=0}^{L} Q_\ell(\boldsymbol{\Delta}_\ell; G(u_\ell - u_{\ell-1})) \;=\; T_1 + T_2(\boldsymbol{\Delta}_*) \;,$$

where

$$T_1 \;:=\; I(G(u)) - \sum_{\ell=0}^{L} I_\ell(G(u_\ell - u_{\ell-1})) \;, \qquad T_2(\boldsymbol{\Delta}_*) \;:=\; \sum_{\ell=0}^{L} (I_\ell - Q_\ell(\boldsymbol{\Delta}_\ell))(G(u_\ell - u_{\ell-1})) \;.$$

We have

$$(I(G(u)) - Q_*^L(\boldsymbol{\Delta}_*; G(u))^2 \;=\; T_1^2 + 2\, T_1 \cdot T_2(\boldsymbol{\Delta}_*) + (T_2(\boldsymbol{\Delta}_*))^2 \;,$$

and the expectation with respect to $\boldsymbol{\Delta}_* \in [0,1]^{s_*}$ is

$$\mathbb{E}[(I(G(u)) - Q_*^L(\cdot; G(u))^2] \;=\; T_1^2 + 2\, T_1 \cdot \mathbb{E}[T_2] + \mathbb{E}[T_2^2] \;.$$

ML-QMC-FE Error Analysis

$$\mathbb{E}[T_2] = \sum_{\ell=0}^{L} \int_{[0,1]^{s_\ell}} ((I_\ell - Q_\ell(\boldsymbol{\Delta}_\ell))(G(u_\ell - u_{\ell-1})))\, \mathrm{d}\boldsymbol{\Delta}_\ell = 0 \,,$$

and

$$\begin{aligned}\mathbb{E}[T_2^2] &= \sum_{\ell=0}^{L} \int_{[0,1]^{s_\ell}} ((I_\ell - Q_\ell(\boldsymbol{\Delta}_\ell))(G(u_\ell - u_{\ell-1})))^2\, \mathrm{d}\boldsymbol{\Delta}_\ell \\ &\quad + \sum_{\ell=0}^{L} \int_{[0,1]^{s_\ell}} (I_\ell - Q_\ell(\boldsymbol{\Delta}_\ell))(G(u_\ell - u_{\ell-1}))\, \mathrm{d}\boldsymbol{\Delta}_\ell \\ &\qquad \cdot \sum_{\substack{\ell'=0 \\ \ell'\neq\ell}}^{L} \int_{[0,1]^{s_{\ell'}}} (I_{\ell'} - Q_{\ell'}(\boldsymbol{\Delta}_{\ell'}))(G(u_{\ell'} - u_{\ell'-1}))\, \mathrm{d}\boldsymbol{\Delta}_{\ell'} \\ &= \sum_{\ell=0}^{L} \mathbb{E}[((I_\ell - Q_\ell(\cdot))(G(u_\ell - u_{\ell-1})))^2] \,, \end{aligned} \tag{20}$$

$$\mathbb{E}[(I(G(u)) - Q_*^L(\cdot; G(u))^2] = T_1^2 + \mathbb{E}[T_2^2] \,. \tag{21}$$

ML-QMC-FE Error Analysis

$$\begin{aligned} |T_1| &\leq \sup_{\boldsymbol{y}\in U} |G(u(\cdot,\boldsymbol{y}) - u_{h_L}(\cdot,\boldsymbol{y}))| + \theta_L \sup_{\boldsymbol{y}\in U} |G(u_{h_L}(\cdot,\boldsymbol{y}) - u_{h_L}(\cdot,(\boldsymbol{y}_{\{1,\dots,L\}};0)))| \\ &\leq \sup_{\boldsymbol{y}\in U} |G(u(\cdot,\boldsymbol{y}) - u_{h_L}(\cdot,\boldsymbol{y}))| + \theta_L \|G(\cdot)\|_{V^*} \sup_{\boldsymbol{y}\in U} \|u_{h_L}(\cdot,\boldsymbol{y}) - u_{h_L}(\cdot,(\boldsymbol{y}_{\{1,\dots,L\}};0))\|_V \\ &\leq C\, h_L^2 \, \|f\|_{L^2(D)} \, \|G(\cdot)\|_{L^2(D)} + \theta_L \frac{\|f\|_{V^*}\, \|G(\cdot)\|_{V^*}}{2\, a_{\min}} \sum_{j\geq s_L+1} b_j \ , \end{aligned} \tag{22}$$

$$\mathbb{E}[T_2^2] \leq \sum_{\ell=0}^{L} \left(\sum_{\emptyset\neq\mathfrak{u}\subseteq\{1:s_\ell\}} \gamma_{\mathfrak{u}}^{\lambda} \, [\rho(\lambda)]^{|\mathfrak{u}|} \right)^{1/\lambda} (N_\ell - 1)^{-1/\lambda} \, \|G(u_{h_\ell}^{s_\ell} - u_{h_{\ell-1}}^{s_{\ell-1}})\|_{\mathcal{W}_{s_\ell,\boldsymbol{\gamma}}}^2 \ . \tag{23}$$

To estimate each term in (23) for $\ell \neq 0$, we write

$$\|G(u_{h_\ell}^{s_\ell} - u_{h_{\ell-1}}^{s_{\ell-1}})\|_{\mathcal{W}_{s_\ell,\boldsymbol{\gamma}}} \leq \|G(u_{h_\ell}^{s_\ell} - u_{h_{\ell-1}}^{s_\ell})\|_{\mathcal{W}_{s_\ell,\boldsymbol{\gamma}}} + \|G(u_{h_{\ell-1}}^{s_\ell} - u_{h_{\ell-1}}^{s_{\ell-1}})\|_{\mathcal{W}_{s_\ell,\boldsymbol{\gamma}}} \ . \tag{24}$$

ML-QMC-FE Error Analysis

Theorem (Parametric regularity of PDE solution)

(a) Under Assumptions $(\mathbf{A1})$ and $(\mathbf{A2})$, for every $f \in V^*$, every $\boldsymbol{y} \in U$ and every $\boldsymbol{\nu} \in \mathfrak{F}$, the solution $u(\cdot, \boldsymbol{y})$ of the parametric weak problem (5) satisfies

$$\left\| \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} u(\cdot, \boldsymbol{y}) \right\|_V \leq |\boldsymbol{\nu}|! \left(\prod_{j \geq 1} b_j^{\nu_j} \right) \frac{\|f\|_{V^*}}{a_{\min}} , \tag{25}$$

where b_j is as defined in (9).

(b) If, in addition, $f \in L^2(D)$ and if Assumption $(\mathbf{A4})$ holds, then for every $\kappa \in (0, 1]$ there holds

$$\left\| \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} u(\cdot, \boldsymbol{y}) \right\|_Z \leq C\, |\boldsymbol{\nu}|! \left(\prod_{j \geq 1} \bar{b}_j^{\nu_j} \right) \|f\|_{L^2(D)} , \tag{26}$$

where

$$\bar{b}_j := b_j + \kappa \left(\|\nabla \psi_j\|_{L^\infty(D)} + B\, \|\psi_j\|_{L^\infty(D)} \right) , \qquad j \geq 1 , \tag{27}$$

with

$$B := \frac{1}{a_{\min}} \sup_{\boldsymbol{z} \in U} \|\nabla a(\cdot, \boldsymbol{z})\|_{L^\infty(D)} < \infty . \tag{28}$$

ML-QMC-FE Error Analysis

Theorem

a) Under Assumptions $(\mathbf{A1})$, $(\mathbf{A2})$, $(\mathbf{A4})$, and $(\mathbf{A6})$, for every $f \in L^2(D)$, every $G(\cdot) \in L^2(D)$, every $\kappa \in (0,1]$, and every $s \in \mathbb{N}$, we have

$$\|G(u^s - u_h^s)\|_{\mathcal{W}_{s,\boldsymbol{\gamma}}} \\ \leq C\, h^2\, a_{\max}\, \|f\|_{L^2(D)}\, \|G(\cdot)\|_{L^2(D)} \left(\sum_{\mathfrak{u} \subseteq \{1,\ldots,s\}} \frac{((|\mathfrak{u}|+1)!)^2 \prod_{j\in\mathfrak{u}} \bar{b}_j^2}{\gamma_{\mathfrak{u}}} \right)^{1/2},$$

where $\bar{b}_j$ is as defined in (27), and the constant $C > 0$ is independent of κ and s.

b) Under Assumptions $(\mathbf{A1})$ and $(\mathbf{A2})$, for every $f \in V^*$, every $G(\cdot) \in V^*$, every $h > 0$, and every $\ell \geq 1$,

$$\|G(u_h^{s_\ell} - u_h^{s_{\ell-1}})\|_{\mathcal{W}_{s_\ell,\boldsymbol{\gamma}}} \leq \frac{\|f\|_{V^*}\, \|G(\cdot)\|_{V^*}}{a_{\min}} \left(\sum_{\mathfrak{u} \in \mathcal{S}_\ell \setminus \mathcal{S}_{\ell-1}} \frac{(|\mathfrak{u}|!)^2 \prod_{j\in\mathfrak{u}} b_j^2}{\gamma_{\mathfrak{u}}} \right)^{1/2},$$

where b_j is as in (9), and

$$\mathcal{S}_\ell := \{\mathfrak{u} \subseteq \{1,\ldots,s_\ell\}\}\,, \qquad \ell \geq 0\,,$$

with the convention that $\mathcal{S}_{-1} := \emptyset$.

ML-QMC-FE Error Analysis

Simplified root-mean-square error bound:

$$\sqrt{\mathbb{E}[(I(G(u)) - Q_*^L(G(u);\cdot))^2]} \leq \hat{C}\, D_{\boldsymbol{\gamma}}(\lambda)\, \frac{a_{\max}}{a_{\min}}\, \|f\|_{L^2(D)}\, \|G(\cdot)\|_{L^2(D)} \left(h_L^2 + \sum_{\ell=0}^{L} N_\ell^{-1/(2\lambda)}\, h_\ell^2 \right), \tag{29}$$

where

$$D_{\boldsymbol{\gamma}}(\lambda) := \left(\sum_{|\mathfrak{u}|<\infty} \gamma_{\mathfrak{u}}^{\lambda}\, [\rho(\lambda)]^{|\mathfrak{u}|} \right)^{1/2\lambda} \left(\sum_{|\mathfrak{u}|<\infty} \frac{((|\mathfrak{u}|+1)!)^2 \prod_{j\in\mathfrak{u}} \bar{b}_j^2}{\gamma_{\mathfrak{u}}} \right)^{1/2}. \tag{30}$$

Choice of QMC-weight vector $\boldsymbol{\gamma}$?

ML-QMC-FE Error Analysis

With $\bar{b}_j$ defined as in (27) for fixed $\kappa \in (0,1]$, suppose that

$$\sum_{j\geq 1} \bar{b}_j^q \,<\, \infty \qquad \text{for some} \quad 0 < q \leq 1 \,, \tag{31}$$

and when $q = 1$ assume additionally that

$$\sum_{j\geq 1} \bar{b}_j \,<\, 2 \,. \tag{32}$$

Let

$$\lambda \,=\, \lambda_q \,:=\, \begin{cases} \frac{1}{2-2\delta} & \text{for some} \quad \delta \in (0,1/2) \quad \text{when } q \in (0,2/3] \,, \\ \frac{q}{2-q} & \qquad\qquad\qquad\qquad \text{when } q \in (2/3,1) \,, \\ 1 & \qquad\qquad\qquad\qquad \text{when } q = 1 \,. \end{cases} \tag{33}$$

Then the choice of weights

$$\gamma_{\mathfrak{u}} \,=\, \gamma_{\mathfrak{u}}^* \,:=\, \left((|\mathfrak{u}|+1)! \prod_{j\in\mathfrak{u}} \frac{\bar{b}_j}{\sqrt{\rho(\lambda)}} \right)^{2/(1+\lambda)} \tag{34}$$

minimizes $D_{\boldsymbol{\gamma}}(\lambda)$ given in (30) and ensures that $D_{\boldsymbol{\gamma}}(\lambda) < \infty$.

ML-QMC-FE Error Analysis

Choose N_ℓ such that

$$\sum_{\ell=0}^{L} N_\ell^{-1/(2\lambda)}\, h_\ell^2 \;=\; \mathcal{O}(h_L^2)\;. \tag{35}$$

We shall achieve this by taking, for $\sigma > 0$, $N_\ell^{-1/(2\lambda)}\, 2^{-2\ell} = (\ell+1)^{-(1+\sigma)}\, 2^{-2L}$ for $\ell = 0,\ldots,L$, noting that $\sum_{\ell=0}^{L}(\ell+1)^{-(1+\sigma)} < \zeta(1+\sigma) < \infty$. Equivalently, for $\sigma > 0$ we take

$$N_\ell \;:=\; 2^{4\lambda(L-\ell)}\,(\ell+1)^{2\lambda(1+\sigma)} \quad \text{for} \quad \ell = 0,\ldots,L\;. \tag{36}$$

Then we have

$$\sqrt{\mathbb{E}[(I(G(u)) - Q_*^L(G(u);\cdot))^2]} \;=\; \mathcal{O}\left(2^{-2L}\right) = \mathcal{O}(h_L^2)\;. \tag{37}$$

ML-QMC-FE Error Analysis

Scenario 1.

Orthogonality property between ψ_j and FE basis holds.

Then the cost of the algorithm is independent of s_ℓ (!) and bounded by

$$\mathrm{cost}(Q_*^L) \;=\; \mathcal{O}\left(\sum_{\ell=0}^{L} N_\ell\, M_{h_\ell}\right) \;=\; \mathcal{O}\left(2^{4\lambda L} \sum_{\ell=0}^{L} 2^{(-4\lambda+d)\ell}\,(\ell+1)^{2\lambda(1+\sigma)}\right)$$

$$= \begin{cases} \mathcal{O}\big(2^{4\lambda L}\big) & \text{if } d < 4\lambda\ , \\ \mathcal{O}\big(2^{4\lambda L}\, L^{2\lambda(1+\sigma)+1}\big) & \text{if } d = 4\lambda\ , \\ \mathcal{O}\big(2^{dL}\, L^{2\lambda(1+\sigma)}\big) & \text{if } d > 4\lambda\ . \end{cases}$$

ML-QMC-FE Error Analysis

Scenario 2.

Let $p < q \leq 1$ and assume that the orthogonality property does not hold (e.g Karhunen-Loève expansion).

Then

$$\mathrm{cost}(Q_*^L) \;\leq\; \begin{cases} \mathcal{O}\big(2^{4\lambda L}\big) & \text{if } d < 4\lambda - \frac{4p}{q-p}\ , \\ \mathcal{O}\big(2^{4\lambda L}\, L^{2\lambda(1+\sigma)+1}\big) & \text{if } d = 4\lambda - \frac{4p}{q-p}\ , \\ \mathcal{O}\big(2^{2Lp/(1-p)+4\lambda L}\big) & \text{if } 4\lambda - \frac{4p}{q-p} < d < 4\lambda\ , \\ \mathcal{O}\big(2^{2Lp/(1-p)+4\lambda L}\, L^{2\lambda(1+\sigma)+1}\big) & \text{if } d = 4\lambda\ , \\ \mathcal{O}\big(2^{2Lp/(1-p)+dL}\, L^{2\lambda(1+\sigma)}\big) & \text{if } d > 4\lambda\ . \end{cases}$$

Conclusion

- parametric, elliptic PDEs domain $D \subset \mathbb{R}^d$ with ∞-dimensional parameter vector $\boldsymbol{y} \in U = [-1/2, 1/2]^{\mathbb{N}}$
- expectation of $F(\boldsymbol{y}) = G(u(\cdot, \boldsymbol{y}))$ as ∞-dimensional integral
- QMC Quadrature of functionals $G(\cdot)$ of parametric solution $u(\cdot, \boldsymbol{y})$
- randomly shifted QMC lattice rules with N points give same convergence rates as best N-term approximations
- detailed work vs. complexity analysis. For space dimension $d = 2$, with piecewise linear FEM on mesh of meshwidth h in D,
  MLQMC-FEM gives optimal rate $O(h^2)$ with work/memory of *one multigrid FE solution of the parametric, deterministic problem*.
- MLQMC-FEM for other models of randomness (logNormal)