Efficient rare-event simulation for sums of dependent random variables

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joint work with José Blanchet

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   - Sums with general dependent heavy tails
Outline

1. Introduction
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2. Efficient simulation of sums of dependent random variables
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Rare Events

Indexed family of events \( \{A_u : u \in \mathbb{R}\} \) with

\[
P(A_u) \to 0, \quad u \to \infty.
\]

Probabilities difficult to estimate.

Algorithm (Estimator)

Indexed set of simulatable random variables \( \{Z_u : u \in \mathbb{R}\} \) with

\[
\mathbb{E}(Z_u) = P(A_u)
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Efficiency

Efficient Algorithms

- **Logarithmic Efficiency:**

  \[
  \lim_{u \to u_0} \frac{\text{Var}[Z_u]}{P^2 - \epsilon(A_u)} = 0, \quad \forall \epsilon > 0.
  \]

- **Bounded relative error:**

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  \lim_{u \to \infty} \frac{\text{Var}[Z_u]}{P^2(A_u)} < \infty.
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Why?

Approximated confidence interval for an MC estimator $\hat{Z}_u$ is

$$\hat{Z}_u \pm \Phi(1 - \alpha/2) \sqrt{\frac{\text{Var} \, Z_u}{R}}$$

To keep the interval proportional to $\mathbb{P}(A_u)$ we require

$$R \approx \frac{\text{Var} \, \hat{Z}_u}{\mathbb{P}^2(A_u)}.$$
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How?

Variance reduction techniques.
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Tail probabilities of sums

Tail probability of a sum
Let $X_1, \ldots, X_n$. Tail probability of the sum

$$\mathbb{P}(X_1 + \cdots + X_n > u), \quad u \to \infty.$$  

Common Fact: i.i.d. case.
An importance sampling algorithm with exponential change of measure

$$F_\theta(dx) := e^{-\theta x - \kappa(\theta)} F(dx)$$

produces an efficient algorithm if $\theta$ is such that $\mathbb{E}_\theta[X] = u/n$.  

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Right tail probabilities of sums

**Fact**

Severe difficulties occur in the construction of efficient algorithms in presence of heavy tails (Asmussen et al., 2000).

**Heavy Tails**

The Laplace transform (mgf) is not defined for a heavy-tailed random variable $X$. 
Right tail probabilities of sums

Fact

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Heavy Tails

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Heavy-tailed independent random variables

- Asmussen and Kroese (2006). A refined version which is proved to be efficient in the Lognormal and Weibull case.
- Dupuis et al. (2006). Importance sampling algorithm for regularly varying distributions and based on mixtures.
Literature Review

Our contribution: Non-independent case (this talk)


Related papers

Our contribution: Non-independent case (this talk)


Related papers

Another interesting problem

Current work with S. Asmussen and J.L. Jensen.

Efficient estimation of

$$\mathbb{P}(X_1 + \cdots + X_n < nx), \quad x \to 0.$$  

- Exponential Twisting.
- Main difficulty: approximate the Laplace transform.
- Bounded relative error.
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Exploratory analysis

Correlated Lognormals

Normal density contour

Lognormal density contour
Asmussen et al. (2009)

Use the importance sampling distribution

$$\text{LN}(\mu, \delta(x)\Sigma)$$

where $\delta(x)$ is the scaling function.

1. Under very mild conditions of $\delta(x)$: logarithmically efficient.
2. Cross-entropy selection: excellent numerical results
Introduction
Efficient Simulation of Sums
References

Scaled variance algorithm

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Elliptical Distributions

**Definition**

$X$ is elliptical, denoted $E(\mu, \Sigma)$, if

$$X \overset{d}{=} \mu + RC \Theta.$$

- **Location**: $\mu \in \mathbb{R}^n$.
- **Dispersion**: $\Sigma = CtC$ with $C \in \mathbb{R}^{n \times k}$.
- **Spherical**: $\Theta$ uniform random vector on the unit spheroid.
- **Radial**: $R$ positive random variable.
- $R$ and $\Theta$ independent of each other.
Elliptical Distributions

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\( \mathbf{X} \) is elliptical, denoted \( E(\mu, \Sigma) \), if

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- **Location**: \( \mu \in \mathbb{R}^n \). **Dispersion**: \( \Sigma = \mathbf{C}^t \mathbf{C} \) with \( \mathbf{C} \in \mathbb{R}^{n \times k} \).
- **Spherical**: \( \Theta \) uniform random vector on the unit spheroid.
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- \( \mathbf{R} \) and \( \Theta \) independent of each other.
Elliptical Distributions

Example

Logelliptical distributions
- Multivariate Normal.
- Normal Mixtures.
- Symmetric Generalized Hyperbolic Distributions: Hyperbolic Distributions, Multivariate Normal Inverse Gaussian (NIG), Generalized Laplace, Bessel or Symmetric Variance-Gamma, Multivariate $t$.

The Symmetric Generalized Hyperbolic Distributions offer better adjustments than the multivariate normal distributions in financial applications (McNeil et al., 2005).
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Heavy Tails and Log-elliptical distributions

Log-elliptical Distributions

The exponential transformation (component-wise) of an elliptical random vector is known as *logelliptical*. Commonly the marginals are dependent heavy-tailed random variables.

Example

Sum of Logellipticals

\[ G(r, \theta) = \sum_{i=1}^{d} \exp (\mu_i + r\langle A_i, \theta \rangle) . \]
Conditional Monte Carlo

Estimator

An unbiased estimator of $\mathbb{P}(G(R, \Theta) > u)$ is

$$\mathbb{P}(G(R, \Theta) > u | \Theta).$$

Algorithm

- Simulate $\Theta$.
- Determine $B_\Theta := \{r > 0 : G(r, \Theta) > u\}$.
- Return $\mathbb{P}(R \in B_\Theta)$. 
Conditional Monte Carlo

**Estimator**

An unbiased estimator of \( \mathbb{P}(G(R, \Theta) > u) \) is

\[
\mathbb{P}(G(R, \Theta) > u | \Theta).
\]

**Algorithm**

1. Simulate \( \Theta \).
2. Determine \( B_\Theta := \{ r > 0 : G(r, \Theta) > u \} \).
3. Return \( \mathbb{P}(R \in B_\Theta) \).
Case 1: Sums of logelliptical random variables

Logarithmic efficient if

$$\lim_{r \to \infty} \frac{r f_R(r)}{\mathbb{P}(R > r)^{1-\varepsilon}} = 0 \quad \forall \varepsilon > 0.$$ 

$f_R$ is the density of $F$
Efficient for more general functions

Conditions

- $G$ is continuous in the two variables and differentiable in $r$.
- There exists $\delta_0 > 0$, $s_* \in S_d$, $r_0 > 0$ and $\nu > 0$ such that for all $0 < \delta \leq \delta_0$ and all $r > r_0$ it holds

$$\sup_{\theta \in S} G(r, \theta)^{1-\nu \delta} \leq \inf_{\theta \in \mathcal{D}(\delta, s^*)} G(r, \theta)$$

$$\sup_{\theta \in \mathcal{D}(\delta_0, s^*)} G(r, \theta) = \sup_{\theta \in S_d} G(r, \theta),$$

where $\mathcal{D}(\delta, s^*) = \{\theta \in S_d : \|\theta - s^*\| < \delta\}$.
- $0 < \delta_1 < 1$ chosen such that for all $r > r_0$ and $\theta \in \mathcal{D}(\delta, s^*)$ it holds

$$\delta_1 \leq \frac{d \log G(r, \theta)}{dr} \leq \frac{1}{\delta_1}.$$
Conditional Monte Carlo

Interesting cases where we have proved efficiency

Sums, maxima, norms and portfolios of options with Symmetric Generalized Hyperbolic Distributions.
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Exploratory analysis
Auxiliary IS distribution

- Set $b = \log(x/n)$.
- Take the distribution $G_b$ of any efficient IS algorithm for

$$\mathbb{E} \left[ \sum_{i=1}^{d} I(Y_i > b) \right].$$

- Use $G_b$ as an IS distribution for estimating

$$\mathbb{P}(e^{Y_1} + \cdots + e^{Y_n} > x).$$
Efficiency

The last algorithm is efficient if

\[
\lim_{b \to \infty} \frac{\log \mathbb{P}(Y_i > b - c)}{\log \mathbb{P}(Y_i > b)} = 1.
\]

We say $Y_i$ are Logarithmically Long Tailed.

Observation

This condition includes the most practical heavy-tailed distributions.
Efficiency

The last algorithm is efficient if

\[ \lim_{b \to \infty} \frac{\log \mathbb{P}(Y_i > b - c)}{\log \mathbb{P}(Y_i > b)} = 1. \]

We say \( Y_i \) are \textit{Logarithmically Long Tailed}.

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This condition includes the most practical heavy-tailed distributions.
Thanks!


