Efficient rare-event simulation for sums of dependent random variables

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joint work with José Blanchet

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MCQMC
UNSW, Sydney, Australia

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   - Rare Event Simulation
   - Efficient Simulation for Sums of Random Variables
2. Efficient simulation of sums of dependent random variables
   - Scaled variance for sums of correlated lognormals
   - Conditional Monte Carlo for functions of logellipticals
   - Sums with general dependent heavy tails

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Outline

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Rare Events

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Indexed family of events $\{A_u : u \in \mathbb{R}\}$ with

$$\mathbb{P}(A_u) \to 0, \qquad u \to \infty.$$

Probabilities difficult to estimate.

Algorithm (Estimator)

Indexed set of simulatable random variables $\{Z_u : u \in \mathbb{R}\}$ with

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Efficiency

Efficient Algorithms

- Logarithmic Efficiency:

$$\lim_{u \to u_0} \frac{\mathbb{V}\mathrm{ar}\,[Z_u]}{\mathbb{P}^{2-\epsilon}(A_u)} = 0, \qquad \forall \epsilon > 0.$$

- Bounded relative error:

$$\lim_{u \to \infty} \frac{\mathbb{V}\mathrm{ar}\,[Z_u]}{\mathbb{P}^{2}(A_u)} < \infty.$$

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Why?

Approximated confidence interval for an MC estimator $\widehat{z}_u$ is

$$\widehat{z}_u \pm \Phi(1-\alpha/2)\sqrt{\frac{\mathbb{V}\mathrm{ar}\, Z_u}{R}}$$

To keep the interval proportional to $\mathbb{P}(A_u)$ we require

$$R \approx \frac{\mathbb{V}\mathrm{ar}\, \widehat{z}_u}{\mathbb{P}^2(A_u)}.$$

How?

Variance reduction techniques.

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Tail probabilities of sums

Tail probability of a sum

Let $X_1, \ldots, X_n$. Tail probability of the sum

$$\mathbb{P}(X_1 + \cdots + X_n > u), \qquad u \to \infty.$$

Common Fact: i.i.d. case.

An importance sampling algorithm with exponential change of measure

$$F_\theta(dx) := \mathrm{e}^{-\theta x - \kappa(\theta)} F(dx)$$

produces an efficient algorithm if θ is such that $\mathbb{E}_\theta[X] = u/n$.

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Right tail probabilities of sums

Fact

Severe difficulties occur in the construction of efficient algorithms in presence of heavy tails (Asmussen et al., 2000).

Heavy Tails

The Laplace transform (mgf) is not defined for a heavy-tailed random variable X.

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Heavy Tails

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Literature Review

Heavy-tailed independent random variables

- Asmussen and Binswanger (1997). First efficient algorithm for regularly varying distribution.
- Asmussen and Kroese (2006). A refined version which is proved to be efficient in the Lognormal and Weibull case.
- Boots and Shahabuddin (2001) y Juneja and Shahabuddin (2002). Importance sampling (hazard rate).
- Dupuis et al. (2006). Importance sampling algorithm for regularly varying distributions and based on mixtures.
- Blanchet and Li (2011). Adaptive method for subexponential random variables.

Literature Review

Our contribution: Non-independent case (this talk)

- Blanchet et al. (2008). Strategies for Log-elliptical random variables.
- Asmussen et al. (2009). Considers correlated lognormal random variables.
- Blanchet and Rojas-Nandayapa (2012) Improved algorithm for tail probabilities of log-elliptics. An additional more general algorithm for sums.

Related papers

Klöppel et al. (2010) y Chan and Kroese (2009). Similar strategies.

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Another interesting problem

Current work with S. Asmussen and J.L. Jensen.

Efficient estimation of

$$\mathbb{P}(X_1 + \cdots + X_n < nx), \qquad x \to 0.$$

- Exponential Twisting.
- Main difficulty: approximate the Laplace transform.
- Bounded relative error.

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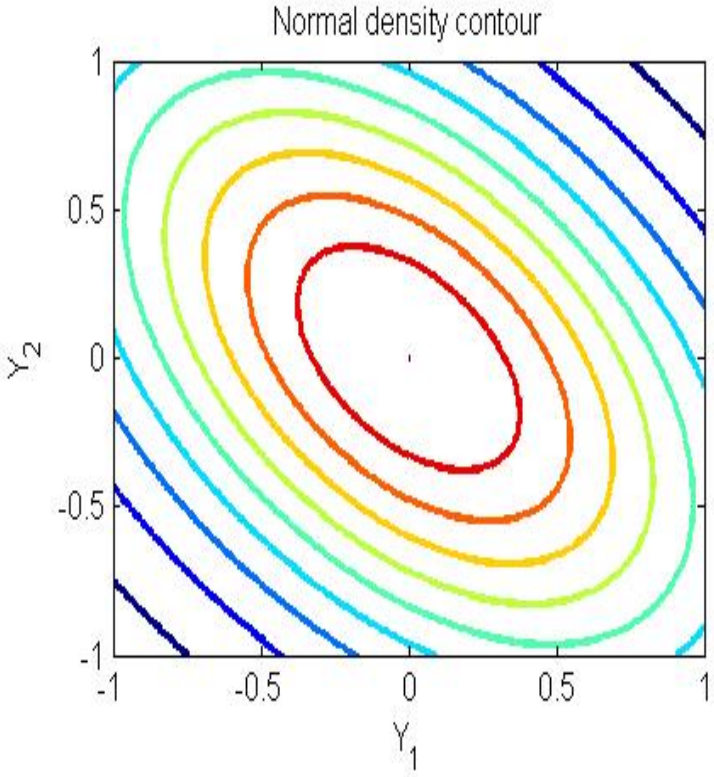
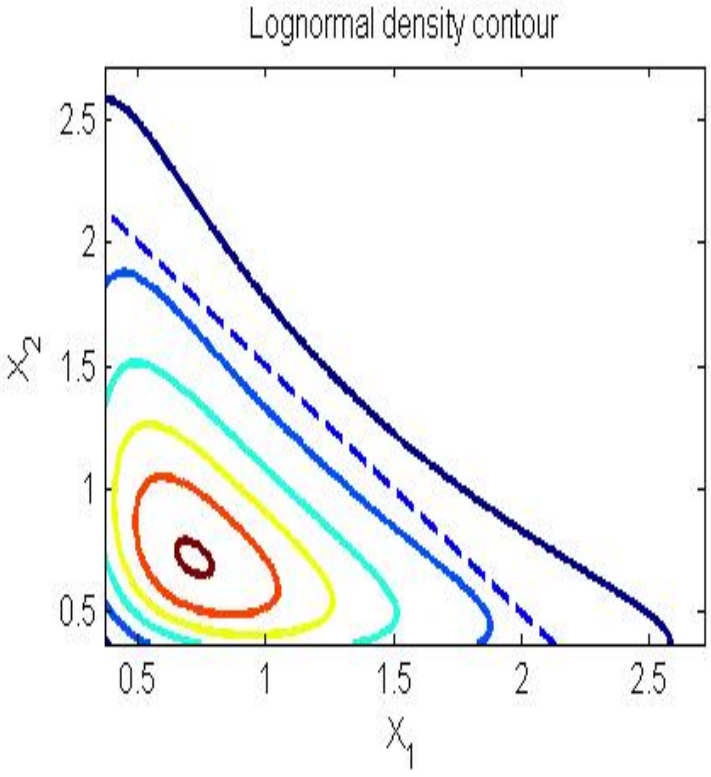
Exploratory analysis

Correlated Lognormals





Scaled variance algorithm

Asmussen et al. (2009)

Use the importance sampling distribution

$$\mathrm{LN}(\mu, \delta(x)\Sigma)$$

where $\delta(x)$ is the scaling function.

1. Under very mild conditions of $\delta(x)$: logarithmically efficient.
2. Cross-entropy selection: excellent numerical results

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Elliptical Distributions

Definition

$\mathbf{X}$ is elliptical, denoted $E(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, if

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + R\,\mathbf{C}\,\Theta.$$

- **Location**: $\boldsymbol{\mu} \in \mathbb{R}^n$. **Dispersion**: $\boldsymbol{\Sigma} = \mathbf{C}^t\mathbf{C}$ with $\mathbf{C} \in \mathbb{R}^{n\times k}$.
- **Spherical**: Θ uniform random vector on the unit spheroid. **Radial**: R positive random variable.
- R and Θ independent of each other.

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Elliptical Distributions

Example

Logelliptical distributions

- Multivariate Normal.
- Normal Mixtures.
- Symmetric Generalized Hyperbolic Distributions: Hyperbolic Distributions, Multivariate Normal Inverse Gaussian (NIG), Generalized Laplace, Bessel or *Symmetric Variance-Gamma*, Multivariate t.

The *Symmetric Generalized Hyperbolic Distributions* offer better adjustments than the multivariate normal distributions in financial applications (McNeil et al., 2005).

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Heavy Tails and Log-elliptical distributions

Log-elliptical Distributions

The exponential transformation (component-wise) of an elliptical random vector is known as *logelliptical*. Commonly the marginals are dependent heavy-tailed random variables.

Example

Sum of Logellipticals

$$G(r,\theta) = \sum_{i=1}^{d} \exp\left(\mu_i + r\left\langle \mathbf{A}_i, \theta \right\rangle\right).$$

Conditional Monte Carlo

Estimator

An unbiased estimator of $\mathbb{P}(G(R,\Theta) > u)$ is

$$\mathbb{P}(G(R,\Theta) > u|\Theta).$$

Algorithm

- Simulate Θ.
- Determine $B_\Theta := \{r > 0 : G(r,\Theta) > u\}$.
- Return $\mathbb{P}(R \in B_\Theta)$.

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Conditional Monte Carlo

Case 1: Sums of logelliptical random variables

Logarithmic efficient if

$$\lim_{r\to\infty} \frac{r\, f_R(r)}{\mathbb{P}(R > r)^{1-\varepsilon}} = 0 \qquad \forall \varepsilon > 0.$$

f_R is the density of F

Conditional Monte Carlo

Efficient for more general functions

Conditions

- G is continuous in the two variables and differentiable in r.
- There exists $\delta_0 > 0$, $s_* \in \mathcal{S}_d$, $r_0 > 0$ and $v > 0$ such that for all $0 < \delta \leq \delta_0$ and all $r > r_0$ it holds

$$\sup_{\theta \in \mathcal{S}} G(r,\theta)^{1-v\delta} \leq \inf_{\theta \in \mathcal{D}(\delta, s^*)} G(r,\theta)$$

$$\sup_{\theta \in \mathcal{D}(\delta_0, s^*)} G(r,\theta) = \sup_{\theta \in \mathcal{S}_d} G(r,\theta),$$

  where $\mathcal{D}(\delta, s^*) = \{\theta \in \mathcal{S}_d : \|\theta - s^*\| < \delta\}$.
- $0 < \delta_1 < 1$ chosen such that for all $r > r_0$ and $\theta \in \mathcal{D}(\delta, s^*)$ it holds

$$\delta_1 \leq \frac{d \log G(r,\theta)}{dr} \leq \frac{1}{\delta_1}.$$

Conditional Monte Carlo

Interesting cases where we have proved efficiency

Sums, maxima, norms and portfolios of options with Symmetric Generalized Hyperbolic Distributions.

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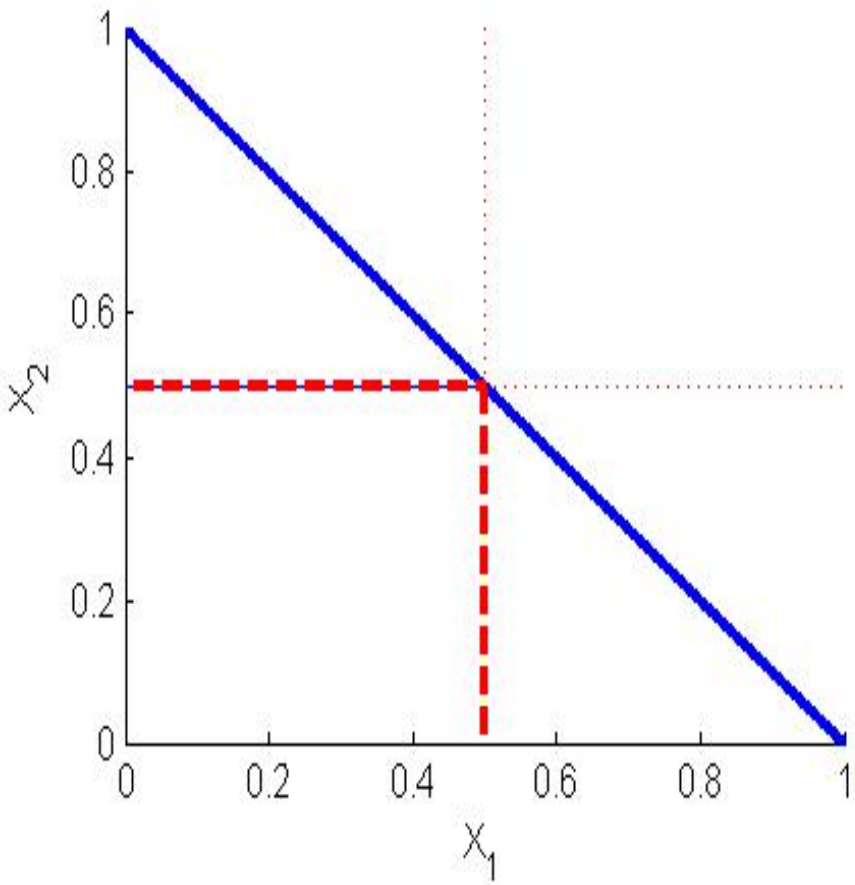

Importance Sampling Distribution

Auxiliary IS distribution

- Set $b = \log(x/n)$.
- Take the distribution G_b of any efficient IS algorithm for

$$\mathbb{E}\left[\sum_{i=1}^{d} I(Y_i > b)\right].$$

- Use G_b as an IS distribution for estimating $\mathbb{P}(\mathrm{e}^{Y_1} + \cdots + \mathrm{e}^{Y_n} > x)$.

Efficiency

Efficiency

The last algorithm is efficient if

$$\lim_{b\to\infty} \frac{\log \mathbb{P}\left(Y_i > b - c\right)}{\log \mathbb{P}\left(Y_i > b\right)} = 1.$$

We say Y_i are *Logarithmically Long Tailed*.

Observation

This condition includes the most practical heavy-tailed distributions.

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Thanks!

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