Calculation of the intermediate bound on the star discrepancy

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Let \( P_n(z) = \{ jz/n \}, j = 0, \ldots, n - 1 \) denote the \( n \)-point set of a rank-1 lattice rule with generating vector \( z \in \mathbb{Z}^d \).

Then the star discrepancy of this point set is given by

\[
D^*(P_n(z)) := \sup_{x \in [0,1)^d} \left| \text{discr}(x, P_n) \right|
\]

where \( \text{discr}(x, P_n) \) is the ‘local discrepancy’ given by

\[
\text{discr}(x, P_n) = \frac{|P_n(z) \cap [0, x)|}{n} - \text{Vol}([0, x)).
\]
At the previous MCQMC conference, an intermediate bound on the star discrepancy was introduced:

$$D^*(P_n(z)) \leq \frac{d}{n} + T(z, n) \leq \frac{d}{n} + W(z, n) \leq \frac{d}{n} + \frac{1}{2} R(z, n).$$

Though values of bound with $T(z, n)$ are better than with $R(z, n)/2$, calculation of $T(z, n)$ requires $O(n^2 d)$ operations compared to $O(nd)$ for $R(z, n)$.

Introduction of $W(z, n)$ resulted in bounds close to those obtained with $T(z, n)$, but with the operation count similar to that required to calculate $R(z, n)$.

CBC construction based on $W(z, n)$ results in $O(n^{-1} (\ln(n))^d)$ bound on star discrepancy for $n$ prime.
This $W(z, n)$ is the lattice rule quadrature error in approximating the integral of $\prod_{i=1}^{d} F_n(x_i)$, where

$$F_n(x) = 1 + \frac{1}{n} \sum_{-n/2 < h \leq n/2, h \neq 0} G(|h|/n) e^{2\pi i h x}, \quad x \in [0, 1).$$

Here

$$G(x) = \begin{cases} 
1/(\pi x) + \pi x/6 + 7\pi^3/2880 & \text{for } x \in (0, \kappa], \\
c_1 + c_2 x & \text{for } x \in (\kappa, 1/2],
\end{cases}$$

where $\kappa = 0.46$, with $c_1$ and $c_2$ chosen so that $G$ is continuous at $x = \kappa$ and $G(1/2) = 1$.

Then $c_1 \approx 1.102449$, and $c_2 \approx -0.204898$. 
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$1/\sin(\pi x)$

$G(x)$

$1/(2^x)$

Graph showing the functions $1/\sin(\pi x)$, $G(x)$, and $1/(2^x)$.
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$1/\sin(\pi x)$

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Graph showing functions $1/\sin(\pi x)$, $G(x)$, and $1/(2^x)$.
Calculating $F_n$

Since components of points of a rank-1 lattice rule are of the form $j/n$ for $j = 0, \ldots, n - 1$, we need the values of

$$F_n(j/n) = 1 + \frac{1}{n} \sum_{-n/2 < h \leq n/2 \atop h \neq 0} G(|h|/n) e^{2\pi i h j/n}, \quad j = 0, \ldots, n - 1.$$ 

For simplicity, let us assume $n$ odd.

Because of symmetry, we may assume $j = 0, \ldots, \lfloor n/2 \rfloor$. 
Then we may write

\[ F_n(x) = 1 + 2S_n(x), \]

where

\[ S_n(x) = \frac{1}{n} \sum_{h=1}^{(n-1)/2} G(h/n) \cos(2\pi hx). \]

Now let \( \alpha(n) = \lfloor \kappa n \rfloor + 1. \)

Then \( 0 < h/n \leq \kappa \) for \( h = 1, \ldots, \alpha(n) - 1 \) and \( \kappa < h/n < 1/2 \) for \( h = \alpha(n), \ldots, (n - 1)/2. \)
So we may write

\[ S_n(x) = \frac{1}{\pi} \sum_{h=1}^{\alpha(n)-1} \frac{\cos(2\pi hx)}{h} + \frac{\pi}{6n^2} \sum_{h=1}^{\alpha(n)-1} h \cos(2\pi hx) \]

\[ + \frac{7\pi^3}{2880n} \sum_{h=1}^{\alpha(n)-1} \cos(2\pi hx) \]

\[ + \sum_{h=\alpha(n)}^{(n-1)/2} \left( \frac{c_1}{n} + \frac{c_2 h}{n^2} \right) \cos(2\pi hx). \]

(This last sum is taken to be an empty sum of zero when \( n \) is odd and less than 13.)
Closed form expressions may be obtained for all the sums except the first one:

For integer $m \geq 2$ and $x \in (0, 1)$, we have

$$\sum_{h=1}^{m-1} \cos(2\pi hx) = \frac{\sin(m\pi x) \cos((m - 1)\pi x)}{\sin(\pi x)} - 1 := \sigma_1(x, m).$$

For the case $x = 0$, we set $\sigma_1(0, m) = m - 1$.

For integer $m \geq 2$ and $x \in (0, 1)$,

$$\sum_{h=1}^{m-1} h \cos(2\pi hx) = \frac{m \sin((2m - 1)\pi x)}{2 \sin(\pi x)} - \frac{1 - \cos(2m\pi x)}{4 \sin^2(\pi x)} := \sigma_2(x, m).$$

For the case $x = 0$, we set $\sigma_2(0, m) = (m - 1)m/2$. 
We may then write

\[
S_n(x) = \frac{1}{\pi} \sum_{h=1}^{\alpha(n)-1} \frac{\cos(2\pi hx)}{h} + \frac{\pi}{6n^2} \sigma_2(x, \alpha(n)) + \frac{7\pi^3}{2880n} \sigma_1(x, \alpha(n)) \\
+ \frac{c_1}{n} \left[ \sigma_1(x, (n-1)/2) - \sigma_1(x, \alpha(n)) \right] \\
+ \frac{c_2}{n^2} \left[ \sigma_2(x, (n-1)/2) - \sigma_2(x, \alpha(n)) \right].
\]

It is clear that the time-consuming part of the calculation of \( S_n(j/n) \) is in calculating the values

\[
Y(j, \alpha(n)) := \sum_{h=1}^{\alpha(n)-1} \frac{\cos(2\pi hj/n)}{h}, \quad j = 0, \ldots, (n-1)/2.
\]

Now recall that \( F_n(j/n) = 1 + 2S_n(j/n) \).
We may modify the results in Joe and Sloan (1993) and obtain approximations to the values $F_n(j/n), j = 0, \ldots, (n - 1)/2$, such that they have absolute error no more than $\varepsilon$. These results are based on an asymptotic expansion.

If $\ell$ and $L$ are positive integers satisfying

$$2 \leq \ell \leq \left( \frac{6n^2}{\pi^2} \right)^{1/3} \quad \text{and} \quad \frac{4(L + 1)!}{(2\kappa)^{L+2}(\ell - 1)^{L+2}\pi^{L+3}} \leq \varepsilon,$$  \hspace{1cm} \text{(1)}

then to approximate $F(j/n)$ to the required accuracy, $Y(j, \alpha(n))$ should be calculated directly using its definition for $j = 0, \ldots, \ell - 1$. 

When \( j = \ell, \ldots, (n - 1)/2 \), \( Y(j, \alpha(n)) \) should be approximated by \( K(j/n) \), where

\[
K(x) = -\ln (2|\sin(\pi x)|) - \sum_{i=0}^{L} b_i(x) \cos (\pi [(2\alpha(n) + i - 1)x + (i + 1)/2]) .
\]

In this expression, \( b_0(x) = 1 / (2\alpha(n)|\sin(\pi x)|) \) and

\[
b_{i+1}(x) = \frac{-(i + 1)}{2(\alpha(n) + i + 1)|\sin(\pi x)|} b_i(x).
\]
Recall from (1) that $\ell$ and $L$ satisfy

$$2 \leq \ell \leq \left(\frac{6n^2}{\pi^2}\right)^{1/3} \text{ and } \frac{4(L + 1)!}{(2\kappa)^{L+2}(\ell - 1)^{L+2}\pi^{L+3}} \leq \varepsilon.$$

As an example, the first equation here is satisfied with $\ell = 20$ when $n \geq 115$. Then the second equation is satisfied for $\varepsilon = 10^{-16}$ when $L = 15$. If $\varepsilon = 10^{-18}$, then we can take $L = 19$.

Then approximations to all the values $F(j/n)$, $j = 0, \ldots, (n - 1)/2$, may be obtained with an absolute error of at most $\varepsilon$ using

$$O(\ell n) + O(L) \times ((n + 1)/2 - \ell) = O(n) \text{ operations.}$$
Timing results

These calculations of $W(z, n)$ were done on a machine with the following specifications:

Ubuntu 10.10

Linux 2.6.35-31-generic-pae SMP

Intel(R) Core(TM) i5-2300 CPU @ 2.80GHz

The value of $L$ was 19.
### Results for \(d = 10\) in seconds

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### Results for \(d = 20\) in seconds

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